RIGIDITY OF AREA-MINIMIZING FREE BOUNDARY SURFACES IN MEAN CONVEX THREE-MANIFOLDS

LUCAS C. AMBROZIO

Abstract. We prove a local splitting theorem for three-manifolds with mean convex boundary and scalar curvature bounded from below that contain certain locally area-minimizing free boundary surfaces. Our methods are based on those of Micallef and Moraru [12]. We use this local result to establish a global rigidity theorem for area-minimizing free boundary disks. In the negative scalar curvature case, this global result implies a rigidity theorem for solutions of the Plateau problem with length-minimizing boundary.

1. Introduction and statements of the results

Let M be a Riemannian manifold with boundary ∂M. Free boundary minimal submanifolds arise as critical points of the area functional when one restricts to variations that preserve ∂M (but not necessarily leave it fixed). Many beautiful known results about closed minimal surfaces could guide the formulation of analogous interesting questions about free boundary minimal surfaces. In this paper, inspired by the rigidity theorems for area-minimizing closed surfaces proved in [2], [3], [12] and [13], we investigate rigidity of area-minimizing free boundary surfaces in Riemannian three-manifolds.

Schoen and Yau, in their celebrated joint work, discovered interesting relations between the scalar curvature of a three-dimensional manifold and the topology of stable minimal surfaces inside it, which emerge when one uses the second variation formula for the area, the Gauss equation and the Gauss-Bonnet theorem. An example is given by the following

Theorem 1 (Schoen and Yau). *Let M be an oriented Riemannian three-manifold with positive scalar curvature. Then M has no immersed orientable closed stable minimal surface of positive genus.*

Schoen and Yau used this to prove that any Riemannian metric with nonnegative scalar curvature on the three-torus must be flat. More generally, they proved the following theorem (see [14]):

Theorem 2 (Schoen and Yau). *Let M be a compact oriented manifold. If the fundamental group of M contains a subgroup isomorphic to the fundamental group of the two-torus, then any Riemannian metric on M with nonnegative scalar curvature must be flat.*

The author was supported by CNPq-Brazil and FAPERJ.

The hypothesis on the fundamental group implies that there exists a continuous map f from the two-torus to M that induces an injective homomorphism f_* on the fundamental groups. Then the idea is to apply a minimization procedure among maps that induce the same homomorphism f_* in order to obtain an immersed stable minimal two-torus in (M, g) for any Riemannian metric g. Since any non-flat Riemannian metric with nonnegative scalar curvature on a compact M can be deformed to a metric with positive scalar curvature (see [7]), the theorem follows.

In [5], Fischer-Colbrie and Schoen observed that an immersed, two-sided, stable minimal two-torus in a Riemannian three-manifold with nonnegative scalar curvature must be flat and totally geodesic, and conjectured that Theorem 2 would hold if one merely assume the existence of an area-minimizing two-torus. This conjecture was established by Cai and Galloway [3]. More precisely, they proved that if M is a compact Riemannian manifold which contains a two-sided embedded two-torus that minimizes the area in its isotopy class, then M is flat. The fundamental step was the following local result:

Theorem 3 (Cai and Galloway). *If a Riemannian three-manifold with nonnegative scalar curvature contains an embedded, two-sided, locally area-minimizing two-torus Σ, then the metric is flat in some neighborhood of Σ.*

In recent years, some similar results were proven for closed surfaces other than tori under different scalar curvature hypotheses. In particular, we mention the theorems of Bray, Brendle and Neves [2] and Nunes [13].

Theorem 4 (Bray, Brendle and Neves). *Let (M, g) be a three-manifold with scalar curvature greater than or equal to 2. If Σ is an embedded two-sphere that is locally area-minimizing, then Σ has area less than or equal to 4π. Moreover, if equality holds, then Σ with the induced metric g_Σ has constant Gaussian curvature equal to 1 and there is a neighborhood of Σ in M that is isometric to $((-\epsilon, \epsilon) \times \Sigma, dt^2 + g_\Sigma)$.*

Theorem 5 (Nunes). *Let (M, g) be a three-manifold with scalar curvature greater than or equal to -2. If Σ is an embedded, two-sided, locally area-minimizing closed surface with genus $g(\Sigma)$ greater than 1, then Σ has area greater than or equal to $4\pi(g(\Sigma) - 1)$. Moreover, if equality holds, then Σ with the induced metric g_Σ has constant Gaussian curvature equal to -1 and there is a neighborhood of Σ in M that is isometric to $((-\epsilon, \epsilon) \times \Sigma, dt^2 + g_\Sigma)$.*

These local splitting theorems also imply interesting global theorems (see [2] and [13]).

Let us give a sketch of the proof of Theorems 4 and 5. In order to prove the inequalities for the area of the respective Σ in the statements above, one can follow Schoen and Yau, using the stability of Σ, the Gauss equation and the Gauss-Bonnet theorem. When the area of Σ achieves the equality stated in the respective theorems, there are more restrictions on the intrinsic

and extrinsic geometries of Σ (recall Fischer-Colbrie and Schoen remark), which allowed then to construct a foliation of M around Σ by constant mean curvature surfaces (by using the inverse function theorem). The use of foliations by constant mean curvature surfaces in relation to scalar curvature problems has already appeared in the work of Huisken and Yau [6] and Bray [1]. After this point, they prove that the leaves of the foliation have area not greater than that of Σ. This is achieved by very different means in [2] and [13]. Since Σ is area-minimizing, it follows that each leaf is area-minimizing and its area satisfies the equality stated in the respective theorems, an information that can be used to conclude the local splitting of (M, g) around Σ.

An interesting unified approach to Theorems 3, 4 and 5 was provided by Micallef and Moraru [12], also based on foliations by constant mean curvature surfaces. In our paper, we give proofs of analogous local rigidity theorems for free boundary surfaces, based on their methods.

Our setting is the following. Let (M, g) be a Riemannian three-manifold with boundary ∂M. Let R^M denote the scalar curvature of M and $H^{\partial M}$ denote the mean curvature of ∂M (we follow the convention that a unit sphere in $\mathbb{R}^3$ has mean curvature 2 with respect to the outward normal). Let Σ be a compact, connected surface with boundary $\partial\Sigma$. We say that Σ is properly embedded (or immersed) in M if it is embedded (or immersed) in M and $\Sigma \cap \partial M = \partial\Sigma$. We say that such Σ is locally area-minimizing in M if every nearby properly immersed surface has area greater than or equal to the area of Σ. The first variation formula for the area (see the Appendix) implies that an area-minimizing properly immersed surface Σ is minimal and *free boundary*, i.e., Σ meets ∂M orthogonally along $\partial\Sigma$. Furthermore Σ is *free boundary stable*, i.e., the second variation of area is nonnegative for every variation that preserves the boundary ∂M.

When R^M and $H^{\partial M}$ are bounded from below, one can consider the following functional in the space of properly immersed surfaces:

$$I(\Sigma) = \frac{1}{2} \inf R^M |\Sigma| + \inf H^{\partial M} |\partial\Sigma|,$$

where $|\Sigma|$ denotes the area of Σ and $|\partial\Sigma|$ denotes the length of $\partial\Sigma$.

The next proposition gives an upper bound to $I(\Sigma)$ when one assumes that Σ is a free boundary stable minimal surface:

Proposition 6. *Let (M, g) be a Riemannian three-manifold with boundary ∂M. Assume R^M and $H^{\partial M}$ are bounded from below. If Σ is a properly immersed, two-sided, free boundary stable minimal surface, then*

$$I(\Sigma) \leq 2\pi\chi(\Sigma). \tag{1}$$

where $\chi(\Sigma)$ is the Euler characteristic of Σ. Moreover, the equality holds if, and only if, Σ satisfies the following properties:

a) Σ is totally geodesic in M and $\partial\Sigma$ consists of geodesics of ∂M;

b) The scalar curvature R^M is constant along Σ and equal to $\inf R^M$, and

the mean curvature $H^{\partial M}$ *is constant along* $\partial\Sigma$ *and equal to* $\inf H^{\partial M}$*;*

c) $Ric(N,N) = 0$*, and* N *is in the kernel of the shape operator of* ∂M *along* $\partial\Sigma$*, where* N *is the unit normal vector field of* Σ*.*

In particular, a), b) and c) imply that Σ *has constant Gaussian curvature* $\inf R^M/2$ *and* $\partial\Sigma$ *has constant geodesic curvature* $\inf H^{\partial M}$ *in* Σ*.*

Inequality (1) relates the scalar curvature of M, the mean curvature of ∂M and the topology of the free boundary stable Σ, as in Schoen and Yau's Theorem 1. This connection has also been studied by Chen, Fraser and Pang [4].

For further reference, we will call *infinitesimally rigid* any properly embedded, two-sided, free boundary surface Σ in M that satisfies properties a), b) and c).

It is interesting to have in mind the following model situation. In Riemannian three-manifolds of the form $(\mathbb{R}\times\Sigma, dt^2+g_0)$, where (Σ, g_0) is a compact Riemannian surface with constant Gaussian curvature whose boundary has constant geodesic curvature, all the slices $\{t\}\times\Sigma$ satisfy the hypotheses of Proposition 6 and are infinitesimally rigid. They also have two additional properties: they are in fact area-minimizing and each connected component of their boundary has the shortest possible length in its homotopy class inside the boundary of $\mathbb{R}\times\Sigma$.

Given a properly embedded, two-sided, infinitesimally rigid free boundary minimal surface Σ_0, we construct a foliation $\{\Sigma_t\}_{t\in(-\epsilon,\epsilon)}$ around Σ_0 by constant mean curvature free boundary surfaces, and then analyze the behavior of the area of the surfaces Σ_t following the unified approach of [12]. When $\inf H^{\partial M} > 0$ and each component of $\partial\Sigma$ is locally length-minimizing, or when $\inf H^{\partial M} = 0$, we prove that $|\Sigma_0| \geq |\Sigma_t|$ for every $t\in(-\epsilon,\epsilon)$ (maybe for some smaller ϵ). As a consequence, we obtain a local rigidity theorem for area-minimizing free boundary surfaces in Riemannian three-manifolds with mean convex boundary (i.e., $H^{\partial M}\geq 0$):

Theorem 7. *Let* (M,g) *be a Riemannian three-manifold with mean convex boundary. Assume that* R^M *is bounded from below.*

Let Σ *be a properly embedded, two-sided, locally area-minimizing free boundary surface such that* $I(\Sigma) = 2\pi\chi(\Sigma)$*. Assume that one of the following hypotheses holds:*

i) each component of $\partial\Sigma$ *is locally length-minimizing in* ∂M*; or*

ii) $\inf H^{\partial M} = 0$*.*

Then there exists a neighborhood of Σ *in* (M,g) *that is isometric to* $((-\epsilon,\epsilon)\times\Sigma, dt^2+g_\Sigma)$*, where* (Σ, g_Σ) *has constant Gaussian curvature* $\frac{1}{2}\inf R^M$ *and* $\partial\Sigma$ *has constant geodesic curvature* $\inf H^{\partial M}$ *in* Σ*.*

We use this local result to prove some global rigidity theorems.

Let $\mathcal{F}_M$ be the set of all immersed disks in M whose boundaries are curves in ∂M that are homotopically non-trivial in ∂M. If $\mathcal{F}_M$ is non-empty, we

define

$$\mathcal{A}(M,g) = \inf_{\Sigma \in \mathcal{F}_M} |\Sigma| \quad \text{and} \quad \mathcal{L}(M,g) = \inf_{\Sigma \in \mathcal{F}_M} |\partial\Sigma|.$$

Our first global rigidity theorem involves a combination of these geometric invariants.

Theorem 8. *Let (M,g) be a compact Riemannian three-manifold with mean convex boundary. Assume that $\mathcal{F}_M$ is non-empty. Then*

$$\frac{1}{2}\inf R^M \mathcal{A}(M,g) + \inf H^{\partial M}\mathcal{L}(M,g) \leq 2\pi. \tag{2}$$

Moreover, if equality holds, then the universal covering of (M,g) is isometric to $(\mathbb{R}\times\Sigma_0, dt^2+g_0)$, where (Σ_0,g_0) is a disk with constant Gaussian curvature $\inf R^M/2$ and Σ_0 has constant geodesic curvature $\inf H^{\partial M}$ in (Σ_0,g_0).

The case $\inf R^M = 0$ and $\inf H^{\partial M} > 0$, which includes in particular mean convex domains of the Euclidean space, was treated by M. Li (see his preprint [9]). His approach is similar to the one in [2].

Our proof relies on the fact that $\mathcal{A}(M,g)$ can be realized as the area of a properly embedded free boundary minimal disk Σ_0, by a classical result of Meeks and Yau [10]. Since $H^{\partial M} \geq 0$, we can compare the invariant and $I(\Sigma_0)$, and hence inequality (2) follows from Proposition 6. When equality holds, Σ_0 must be infinitesimally rigid, and then we use the local splitting around Σ_0 given by Theorem 7 and a standard continuation argument to obtain the global splitting of the universal covering.

When $\inf R^M$ is negative, we also prove a rigidity theorem for solutions of the Plateau problem, which is an immediate consequence of Theorem 8.

As before, assume that (M,g) is a compact Riemannian manifold with mean convex boundary. Another classical result of Meeks and Yau [11] says that the Plateau problem has a solution in M for any given closed embedded curve in ∂M that bounds a disk.

In particular, by considering solutions of the Plateau problem for homotopically non-trivial curves in ∂M that bound disks and have the shortest possible length among such curves, we prove the following

Theorem 9. *Let (M,g) be a compact Riemannian three-manifold with mean convex boundary such that $\inf R^M = -2$. Assume that $\mathcal{F}_M$ is non-empty.*

If $\hat{\Sigma}$ is a solution to the Plateau problem for a homotopically non-trivial embedded curve in ∂M that bounds a disk and has length $\mathcal{L}(M,g)$, then

$$|\hat{\Sigma}| \geq \inf H^{\partial M}\mathcal{L}(M,g) - 2\pi. \tag{3}$$

Moreover, if equality holds in (3) for some $\hat{\Sigma}$, then the universal covering of (M,g) is isometric to $(\mathbb{R}\times\Sigma_0, dt^2+g_0)$, where (Σ_0,g_0) is a disk with constant Gaussian curvature $\inf R^M/2$ and $\partial\Sigma_0$ has constant geodesic curvature $\inf H^{\partial M}$ in Σ_0.

Acknowledgments. I am grateful to my Ph.D advisor at IMPA, Fernando Codá Marques, for his constant advice and encouragement. I also thank Ivaldo Nunes for enlightening discussions about free boundary surfaces. Finally, I am grateful to the hospitality of the Institut Henri Poincaré, in Paris, where the first drafts of this work were written in October/November 2012. I was supported by CNPq-Brazil and FAPERJ.

2. Infinitesimal rigidity

Inequality (1) follows from the second variation formula of area for free boundary minimal surfaces, the Gauss equation and the Gauss-Bonnet theorem.

Proof of Proposition 6. Let Σ be a properly immersed, two-sided, free boundary stable minimal surface. Since Σ is two-sided, there exists a unit vector field N along Σ that is normal to Σ. Let X be the unit vector field on ∂M that is normal to ∂M and points outside M. Since Σ is free boundary, the unit conormal ν of $\partial\Sigma$ that points outside Σ coincides with X along $\partial\Sigma$.

Recall that $H^{\partial M}$ is the trace of the shape operator ∇X, under our convention. The free boundary hypothesis implies that k, the geodesic curvature of $\partial\Sigma$ in Σ, can be computed as $k = g(T, \nabla_T \nu) = g(T, \nabla_T X)$, where T is a unit vector field tangent to $\partial\Sigma$. In particular,

$$H^{\partial M} = k + g(N, \nabla_N X). \tag{4}$$

The free boundary stability hypothesis means that, for every $\phi \in C^\infty(\Sigma)$,

$$Q(\phi,\phi) = \int_\Sigma |\nabla\phi|^2 - (Ric(N,N) + |B|^2)\phi^2 dA - \int_{\partial\Sigma} g(N, \nabla_N X)\phi^2 dL \geq 0,$$

where B denotes the second fundamental form of Σ. $Q(\phi,\phi)$ is the second variation of area for variations with variational vector field ϕN along Σ (for the general second variation formula, see [15]).

By evaluating Q on the constant function 1, we have the inequalities

$$\begin{aligned} 0 &\geq \int_\Sigma (Ric(N,N) + |B|^2) dA + \int_{\partial\Sigma} g(N, \nabla_N X) dL \\ &= \frac{1}{2}\int_\Sigma (R^M + H^2 + |B|^2) dA - \int_\Sigma K dA - \int_{\partial\Sigma} k dL + \int_{\partial\Sigma} H^{\partial M} dL \\ &\geq \frac{1}{2} \inf R^M |\Sigma| + \inf H^{\partial M} |\partial\Sigma| - 2\pi\chi(\Sigma). \end{aligned}$$

where we used the Gauss equation, equation (4) and the Gauss-Bonnet theorem. This proves inequality (1).

When the equality holds in (1), every inequality above is in fact an equality. One immediately sees that Σ must be totally geodesic, *b*) holds and $Q(1,1) = 0$. By elementary considerations about bilinear forms, $Q(1,1) = 0$ and $Q(\phi,\phi) \geq 0$ for every $\phi \in C^\infty(\Sigma)$ implies $Q(1,\phi) = 0$ for every $\phi \in C^\infty(\Sigma)$. Hence, by choosing appropriately the arbitrary test function ϕ, we conclude that $Ric(N,N) = 0$ and $g(N, \nabla_N X) = 0$.

Since Σ is totally geodesic, $\nabla_T T$ and $\nabla_T X = \nabla_T \nu$ are tangent to Σ. Hence, the geodesic curvature of $\partial\Sigma$ in ∂M given by $g(N, \nabla_T T)$ vanishes, and since $\nabla_T X$ is also orthogonal to X we conclude that $\nabla_T X$ is proportional to T, which means that T and therefore N are eigenvectors of ∇X on $\partial\Sigma$. The second parts of a) and c) follows.

The final statement is just a consequence of the Gauss equation and equation (4). The converse is immediate from the Gauss-Bonnet theorem. $\square$

3. Construction of the foliation

Given a properly embedded infinitesimally rigid surface Σ in M, there are smooth vector fields Z on M such that $Z(p) = N(p)$ $\forall p \in \Sigma$ and $Z(p) \in T_p\partial M$ $\forall p \in \partial M$. We fix $\phi = \phi(x,t)$ the flow of one of these vector fields and α a real number between zero and one.

The next proposition gives a family of constant mean curvature free boundary surfaces around an infinitesimally rigid surface.

Proposition 10. *Let (M, g) be a Riemannian three-manifold with boundary ∂M. Assume R^M and $H^{\partial M}$ are bounded from below. Let Σ be a properly embedded, two-sided, free boundary surface.*

If Σ is infinitesimally rigid, then there exists $\epsilon > 0$ and a function $w : \Sigma \times (-\epsilon, \epsilon) \to \mathbb{R}$ such that, for every $t \in (-\epsilon, \epsilon)$, the set

$$\Sigma_t = \{\phi(x, w(x,t));\ x \in \Sigma\}$$

is a free boundary surface with constant mean curvature $H(t)$. Moreover, for every $x \in \Sigma$ and every $t \in (-\epsilon, \epsilon)$,

$$w(x,0) = 0, \quad \int_\Sigma (w(x,t) - t)\, dA = 0 \quad \textit{and} \quad \frac{\partial}{\partial t} w(x,t)\Big|_{t=0} = 1.$$

In particular, for some smaller ϵ, $\{\Sigma_t\}_{t\in(-\epsilon,\epsilon)}$ is a foliation of a neighborhood of $\Sigma_0 = \Sigma$ in M.

Proof. As in the proof of Proposition 6, let N denote the unit normal vector field of Σ, and let X denote the unit normal vector field of ∂M that coincides with the exterior conormal ν of $\partial\Sigma$. Let dA be the area element of Σ and let dL be the length element of $\partial\Sigma$.

Given a function u in the Hölder space $C^{2,\alpha}(\Sigma)$, $0 < \alpha < 1$, we consider $\Sigma_u = \{\phi(x, u(x)); x \in \Sigma\}$, which is a properly embedded surface if the norm of u is small enough. We use the subscript u to denote the quantities associated to Σ_u. For example, H_u will denote the mean curvature of Σ_u, N_u will denote the unit normal vector field of Σ_u and X_u will denote the restriction of X to $\partial\Sigma_u$. In particular, $\Sigma_0 = \Sigma$, $H_0 = 0$ (since Σ is totally geodesic) and $g(N_0, X_0) = 0$ (since Σ_0 is free boundary).

Consider the Banach spaces $E = \{u \in C^{2,\alpha}; \int_\Sigma u dA = 0\}$ and $F = \{u \in C^{0,\alpha}; \int_\Sigma u dA = 0\}$. Given small $\delta > 0$ and $\epsilon > 0$, we can define the map

$\Phi : (-\epsilon, \epsilon) \times (B(0,\delta) \subset E) \to F \times C^{1,\alpha}(\partial\Sigma)$ given by

$$\Phi(t,u) = (\ H_{t+u} - \frac{1}{|\Sigma|}\int_\Sigma H_{t+u} dA,\ g(N_{t+u}, X_{t+u})\ ).$$

We claim that $D\Phi_{(0,0)}$ is an isomorphism when restricted to $0 \times E$.

In fact, for each $v \in E$, the map $f : (x,s) \in \Sigma \times (-\epsilon, \epsilon) \mapsto \phi(x, sv(x)) \in M$ gives a variation with variational vector field $\frac{\partial f}{\partial s}|_{s=0} = vZ = vN$ on Σ. Since Σ is infinitesimally rigid we obtain (see Proposition 17 in the Appendix):

$$D\Phi_{(0,0)}(0,v) = \frac{d}{ds}\Big|_{s=0}\Phi(0,sv) = (-\Delta_\Sigma v + \frac{1}{|\Sigma|}\int_{\partial\Sigma}\frac{\partial v}{\partial\nu}dL,\ -\frac{\partial v}{\partial \nu}).$$

The claim follows from classical results for Neumann type boundary conditions for the Laplace operator (see for example [8], page 137).

Now we apply the implicit function theorem: for some smaller ϵ, there exists a function $t \in (-\epsilon,\epsilon) \mapsto u(t) \in B(0,\delta) \subset E$ such that $u(0) = 0$ and $\Phi(t, u(t)) = \Phi(0,0) = (0,0)$ for every t. In other words, the surfaces

$$\Sigma_{t+u(t)} = \{\phi(x, t + u(t)(x));\ x \in \Sigma\}$$

are free boundary constant mean curvature surfaces.

Let $w : (x,t) \in \Sigma \times (-\epsilon,\epsilon) \mapsto t + u(t)(x) \in \mathbb{R}$. By definition, $w(x,0) = u(0)(x) = 0$ for every $x \in \Sigma$ and $w(-,t) - t = u(t)$ belongs to $B(0,\delta) \subset E$ for every $t \in (-\epsilon,\epsilon)$. Observe that the map $G : (x,s) \in \Sigma \times (-\epsilon,\epsilon) \mapsto \phi(x, w(x,s)) \in M$ gives a variation of Σ with variational vector field on Σ given by $\left(\frac{\partial w}{\partial t}|_{t=0}\right) N$. Since for every t we have

$$0 = \Phi(t,u(t)) = (\ H_{w(-,t)} - \frac{1}{|\Sigma|}\int_\Sigma H_{w(-,t)} dA\ ,\ g(N_{w(-,t)}, X_{w(-,t)})\ ),$$

by taking the derivative at $t = 0$ we conclude that $\frac{\partial w}{\partial t}|_{t=0}$ satisfies the homogeneous Neumann problem. Therefore it must be constant on Σ. Since $\int_\Sigma (w(x,t) - t)\, dA = \int_\Sigma u(t)(x) dA = 0$ for every t, by taking again a derivative at $t = 0$ we conclude that $\int_\Sigma \left(\frac{\partial w}{\partial t}|_{t=0}\right) dA = |\Sigma|$. Hence, $\frac{\partial w}{\partial t}|_{t=0} = 1$, as claimed.

Since $G_0(x) = \phi(x,0) = x$, $\partial_t G(x,0) = \frac{\partial w}{\partial t}|_{t=0} N_0(x) = N_0(x)$ for every x in Σ_0 and Σ_0 is properly embedded, by taking a smaller ϵ, if necessary, we can assume that G parametrizes a foliation of M around Σ_0. This finishes the proof of the proposition. □

4. Local Rigidity

We consider a Riemannian three-manifold with mean convex boundary and scalar curvature bounded from below. First we analyze the behavior of the area of surfaces in the family constructed in section 3. This analysis is based on [12].

Proposition 11. *Let (M,g) be a Riemannian three-manifold with mean convex boundary and scalar curvature bounded from below. Let Σ_0 be a properly embedded, two-sided, free boundary, infinitesimally rigid surface.*

Assume that one of the following hypotheses holds:
i) each component of $\partial\Sigma_0$ *is locally length-minimizing in* ∂M*; or*
ii) $\inf H^{\partial M} = 0$.
Let $\{\Sigma_t\}_{t\in(-\epsilon,\epsilon)}$ *be as in Proposition 10. Then*

$$|\Sigma_0| \geq |\Sigma_t|$$

for every $t \in (-\epsilon, \epsilon)$ *(maybe for some smaller* ϵ*).*

Proof. Following the notation of Proposition 10, let $G : \Sigma_0 \times (-\epsilon, \epsilon) \to M$ given by $G_t(x) = \phi(x, w(x, t))$ parametrize the foliation $\{\Sigma_t\}_{t\in(-\epsilon,\epsilon)}$ around the infinitesimally rigid Σ_0. After this point, we will use the subscript t to denote the quantities associated to $\Sigma_t = G_t(\Sigma_0)$.

For each $t \in (-\epsilon, \epsilon)$, the lapse function on Σ_t given by $\rho_t = g(\partial_t G, N_t)$ satisfies the equations (see Proposition 18 in the Appendix):

$$-H'(t) = \Delta_t\rho_t + (Ric(N_t, N_t) + |B_t|^2)\rho_t, \tag{5}$$

$$\frac{\partial\rho_t}{\partial\nu_t} = g(N_t, \nabla_{N_t} X)\rho_t. \tag{6}$$

Furthermore, $\rho_0 = 1$, since $\partial_t G(x, 0) = N_0(x)$ for every $x \in \Sigma$. Hence, we can assume $\rho_t > 0$ for all $t \in (-\epsilon, \epsilon)$. From equation (5) we have

$$H'(t)\frac{1}{\rho_t} = -(\Delta_t\rho_t)\frac{1}{\rho_t} - (Ric(N_t, N_t) + |B_t|^2).$$

Using the Gauss equation, we rewrite

$$H'(t)\frac{1}{\rho_t} = -(\Delta_t\rho_t)\frac{1}{\rho_t} + K_t - \frac{1}{2}(R^M_t + H(t)^2 + |B_t|^2).$$

Recalling that $H(t)$ is constant on Σ_t, we integrate by parts using equation (6) in order to get

$$\begin{aligned} H'(t)\int_\Sigma \frac{1}{\rho_t} dA_t = &-\int_\Sigma \frac{|\nabla_t p_t|^2}{\rho_t^2} dA_t - \int_{\partial\Sigma} g(N_t, \nabla_{N_t} X) dL_t \\ &+ \int_\Sigma K_t dA_t - \frac{1}{2}\int_\Sigma (R^M_t + H(t)^2 + |B_t|^2) dA_t. \end{aligned}$$

Since each Σ_t is free boundary, equation (4) and the Gauss-Bonnet theorem implies

$$\begin{aligned} H'(t)\int_\Sigma \frac{1}{\rho_t} dA_t = &-\int_\Sigma \frac{|\nabla_t p_t|^2}{\rho_t^2} dA_t - \frac{1}{2}\int_\Sigma (R^M_t + H(t)^2 + |B_t|^2) dA_t \\ &- \int_{\partial\Sigma} H^{\partial M}_t dL_t + 2\pi\chi(\Sigma_0). \end{aligned}$$

Finally, since Σ_0 is infinitesimally rigid, the Gauss-Bonnet theorem implies that $I(\Sigma_0) = 2\pi\chi(\Sigma_0)$. Hence, we have the following inequality:

$$\begin{aligned} H'(t)\int_\Sigma \frac{1}{\rho_t}dA_t &\leq I(\Sigma_0) - I(\Sigma_t) \\ &= \frac{1}{2}\inf R^M(|\Sigma_0| - |\Sigma_t|) + \inf H^{\partial M}(|\partial\Sigma_0| - |\partial\Sigma_t|). \end{aligned}$$

By hypothesis, $\inf H^{\partial M} \geq 0$. If each boundary component is locally length-minimizing, the second term in the right hand side is less than or equal to zero, and in case $\inf H^{\partial M} = 0$, it is obviously zero. Therefore

$$H'(t)\int_\Sigma \frac{1}{\rho_t}dA_t \leq \frac{1}{2}\inf R^M(|\Sigma_0| - |\Sigma_t|) = -\frac{1}{2}\inf R^M \int_0^t \frac{d}{ds}|\Sigma_s|ds.$$

Since each Σ_t is free boundary, the first variation formula of area gives

$$\frac{d}{dt}|\Sigma_t| = \int_\Sigma \rho_t H(t)dA_t = H(t)\int_\Sigma \rho_t dA_t. \tag{7}$$

Therefore

$$H'(t)\int_\Sigma \frac{1}{\rho_t}dA_t \leq -\frac{1}{2}\inf R^M \int_0^t H(s)\left(\int_\Sigma \rho_s dA_s\right)ds. \tag{8}$$

Claim: there exists $\epsilon > 0$ such that $H(t) \leq 0$ for every $t \in [0, \epsilon)$.

We consider three cases:

a) $\inf R^M = 0$.
Then it follows immediately from (8) that $H'(t) \leq 0$ for every $t \in [0, \epsilon)$. Since $H(0) = 0$, the claim follows.

b) $\inf R^M > 0$.
Let $\varphi(t) = \int_\Sigma \frac{1}{\rho_t}dA_t$ and $\xi(t) = \int_\Sigma \rho_t dA_t$. Inequality (8) can be rewritten as

$$H'(t) \leq -\frac{1}{2}\inf R^M \frac{1}{\varphi(t)}\int_0^t H(s)\xi(s)ds. \tag{9}$$

By continuity, we can assume that there exists a constant $C > 0$ such that $\frac{1}{\varphi(t)}\int_0^t \xi(s)ds \leq 2C$ for every $t \in [0, \epsilon]$.

Choose $\epsilon > 0$ such that $C \inf R^M \epsilon < 1$. Then $H(t) \leq 0$ for every $t \in [0, \epsilon)$. In fact, suppose that there exists $t_+ \in (0, \epsilon)$ such that $H(t_+) > 0$. By continuity, there exists $t_- \in [0, t_+]$ such that $H(t) \geq H(t_-)$ for every $t \in [0, t_+]$. Notice that $H(t_-) \leq H(0) = 0$. By the mean value theorem, there exists $t_1 \in (t_-, t_+)$ such that $H(t_+) - H(t_-) = H'(t_1)(t_+ - t_-)$. Hence,

since $\inf R^M > 0$, inequality (9) gives

$$\begin{aligned}\frac{H(t_+)-H(t_-)}{t_+-t_-} = H'(t_1) &\leq \frac{1}{2}\inf R^M \frac{1}{\varphi(t_1)}\int_0^{t_1}(-H(s))\xi(s)ds \\ &\leq \frac{1}{2}\inf R^M(-H(t_-))\left(\frac{1}{\varphi(t_1)}\int_0^{t_1}\xi(s)ds\right) \\ &\leq \inf R^M(-H(t_-))C.\end{aligned}$$

It follows that $H(t_+) \leq H(t_-)(1 - C\inf R^M\epsilon)$, which is a contradiction since $H(t_+) > 0$ and $H(t_-) \leq 0$.

c) $\inf R^M < 0$.

Choose $\epsilon > 0$ such that $-C\inf R^M\epsilon < 1$, where $C > 0$ is the same constant that appears in case b). Then $H(t) \leq 0$ for every $t \in [0,\epsilon)$. In fact, suppose that there exists $t_0 \in (0,\epsilon)$ such that $H(t_0) > 0$. Let

$$R = \{t \in [0,t_0];\ H(t) \geq H(t_0)\}.$$

Let $t^* \in [0,\epsilon]$ be the infimum of R. Observe that, by the definition of t^*, $H(t) \leq H(t_0) = H(t^*)$ for every $t \in [0,t^*]$.

If $t^* > 0$, then the mean value theorem implies that there exists $t_1 \in (0,t^*)$ such that $H(t^*) = H'(t_1)t^*$, since $H(0) = 0$. Hence, since $\inf R^M < 0$, inequality (9) gives

$$\begin{aligned}\frac{H(t^*)}{t^*} = H'(t_1) &\leq -\frac{1}{2}\inf R^M \frac{1}{\varphi(t_1)}\int_0^{t_1}H(s)\xi(s)ds \\ &\leq -\frac{1}{2}\inf R^M H(t^*)\left(\frac{1}{\varphi(t_1)}\int_0^{t_1}\xi(s)ds\right) \\ &\leq -\inf R^M H(t^*)C.\end{aligned}$$

It follows that $H(t^*)(1 + C\inf R^M H(t^*)\epsilon) \leq 0$. This is a contradiction since $H(t^*) = H(t_0) > 0$.

Hence, $t^* = 0$, which is again a contradiction since $0 = H(0) \geq H(t_0) > 0$.

This proves the claim. By equation (7), we conclude that $|\Sigma_0| \geq |\Sigma_t|$ for every $t \in [0,\epsilon)$. The proof that $|\Sigma_0| \geq |\Sigma_t|$ for every $t \in (-\epsilon,0]$ is analogous. □

We are now ready to prove the local splitting result, Theorem 7.

Proof of Theorem 7. Since Σ is locally area-minimizing and $I(\Sigma) = 2\pi\chi(\Sigma)$, Σ is infinitesimally rigid. From propositions 10 and 11 we obtain a foliation $\{\Sigma_t\}_{t\in(-\epsilon,\epsilon)}$ around $\Sigma_0 = \Sigma$ such that $|\Sigma_t| \leq |\Sigma_0|$ for every $t \in (-\epsilon,\epsilon)$. Since Σ is locally area-minimizing, each Σ_t is also locally area-minimizing, with $|\Sigma_t| = |\Sigma|$.

It is immediate to see that when $\inf H^{\partial M} = 0$ or when the components

of $\partial\Sigma_0$ are locally length-minimizing,

$$2\pi = I(\Sigma_0) \leq I(\Sigma_t) \leq 2\pi,$$

which implies that each Σ_t is infinitesimally rigid. From equations (5) and (6) in Proposition 11, one sees that for each t the lapse function ρ_t satisfies the homogeneous Neumann problem. Therefore ρ_t is a constant function on Σ_t.

Since we have a foliation, the normal fields of Σ_t define locally a vector field on M. This field is parallel (see [2], [12] or [13]). In particular, its flow is a flow by isometries and therefore provides the local splitting: a neighborhood of Σ_0 is in fact isometric to the product $((-\epsilon,\epsilon)\times\Sigma_0, dt^2+g_{\Sigma_0})$. Since Σ_0 is infinitesimally rigid, (Σ_0, g_{Σ_0}) has constant Gaussian curvature $\inf R^M/2$ and $\partial\Sigma_0$ has constant geodesic curvature $\inf H^M$ in Σ_0.

□

5. Global Rigidity

Before we begin the proofs, we state precisely the result of Meeks and Yau about the existence of area-minimizing free boundary disks that we will use in the sequel (see [10]).

Theorem 12 (Meeks and Yau). *Let (M,g) be a compact Riemannian three-manifold with mean convex boundary. If $\mathcal{F}_M$ is non-empty, then*

1) There exists an immersed minimal disk Σ_0 in M such that $\partial\Sigma_0$ represents a homotopically non-trivial curve on ∂M and $|\Sigma_0| = \mathcal{A}(M,g)$.

2) Any such least area immersed disk is in fact a properly embedded free boundary disk.

We are now ready to prove our main theorems.

Proof of Theorem 8. Since $\mathcal{F}_M$ is non-empty, Theorem 12 says that there exists a properly embedded free boundary minimal disk $\Sigma_0 \in \mathcal{F}_M$ such that $|\Sigma_0| = \mathcal{A}(M,g)$. Since Σ_0 is two-sided and free boundary stable, the inequality follows from Proposition 6:

$$\frac{1}{2}\inf R^M \mathcal{A}(M,g) + \inf H^{\partial M}\mathcal{L}(M,g) \leq I(\Sigma_0) \leq 2\pi.$$

Assume that the equality holds. In case $\inf H^{\partial M}$ is not zero, $\partial\Sigma_0$ must have length $\mathcal{L}(M,g)$, hence it is length-minimizing. In any case, we can apply Theorem 7 to get a local splitting of (M,g) around Σ_0.

Let exp denote the exponential map of (M,g). Let S be the set all $t>0$ such that the map $\Psi : [-t,t]\times\Sigma_0 \to M$ given by $\Psi(s,x) = \exp_x(sN_0(x))$ is well-defined, $\Psi([-t,t]\times\partial\Sigma_0)$ is contained in ∂M and $\Psi : ((-t,t)\times\Sigma_0, ds^2 + g_{\Sigma_0}) \to (M,g)$ is a local isometry.

S is non-empty because of the local splitting. Standard arguments imply that $S = [0,+\infty)$. Therefore we have a well-defined local isometry

$$\Psi : (t,x) \in (\mathbb{R}\times\Sigma_0, dt^2 + g_{\Sigma_0}) \mapsto \exp_x(tN_0(x)) \in (M,g),$$

such that $\Psi(\mathbb{R} \times \partial\Sigma_0)$ is contained in ∂M. Such Ψ is a covering map. This finishes the proof of Theorem 8. $\square$

In order to prove Theorem 9, consider any $\hat{\Sigma}$ as in its statement. $\hat{\Sigma}$ has area at least $\mathcal{A}(M, g)$ and $\partial\hat{\Sigma}$ has length $\mathcal{L}(M, g)$. When $\inf R^M$ is negative,

$$I(\hat{\Sigma}) = \frac{1}{2} \inf R^M |\hat{\Sigma}| + \inf H^{\partial M} |\partial\hat{\Sigma}| \leq \frac{1}{2} \inf R^M \mathcal{A}(M, g) + \inf H^{\partial M} \mathcal{L}(M, g).$$

and therefore Theorem 9 is an immediate corollary of Theorem 8.

Appendix

For completeness we include some general formulae for the infinitesimal variation of some geometric quantities of properly immersed hypersurfaces under variations of the ambient manifold (M^{n+1}, g) that maintain the boundary of the hypersurface inside ∂M.

We begin by fixing some notations. Let (M^{n+1}, g) be a Riemannian manifold with boundary ∂M. Let X denote the unit normal vector field along ∂M that points outside ∂M.

Let Σ^n be a manifold with boundary $\partial\Sigma$ and assume Σ is immersed in M in such way that $\partial\Sigma$ is contained in ∂M. The unit conormal of $\partial\Sigma$ that points outside Σ will be denoted by ν. Given N a local unit normal vector field to Σ, the second fundamental form is the symmetric tensor B on Σ given by $B(U, W) = g(\nabla_U N, W)$ for every U, W tangent to Σ. The mean curvature H is the trace of B. Σ is called *minimal* when $H = 0$ on Σ and *free boundary* when $\nu = X$ on $\partial\Sigma$.

We consider variations of Σ given by smooth maps $f : \Sigma \times (-\epsilon, \epsilon) \to M$ such that, for every $t \in (-\epsilon, \epsilon)$, the map $f_t : x \in \Sigma \mapsto f(x, t) \in M$ is an immersion of Σ in M such that $f_t(\partial\Sigma)$ is contained in ∂M.

The subscript t will be used to denote quantities associated to $\Sigma_t = f_t(\Sigma)$. For example, N_t will denote a local unit vector field normal to Σ_t and H_t will denote the mean curvature of Σ_t.

It will be useful for the computations to introduce local coordinates $x^1, \ldots, x^n$ in Σ. We will also use the simplified notation

$$\partial_t = \frac{\partial f}{\partial t} \quad \text{and} \quad \partial_i = \frac{\partial f}{\partial x_i},$$

where i runs from 1 to n. ∂_t is called the variational vector field. We decompose it in its tangent and normal components:

$$\partial_t = \partial_t^T + v_t N_t,$$

where v_t is the function on Σ_t defined by $v_t = g(\partial_t, N_t)$.

First we look at the variation of the metric tensor $g_{ij} = g(\partial_i, \partial_j)$:

Proposition 13.

$$\begin{aligned} \partial_t g_{ij} &= g(\nabla_{\partial_i} \partial_t, \partial_j) + g(\partial_i, \nabla_{\partial_j} \partial_t), \\ \partial_t g^{ij} &= -2 g^{ik} g^{jl} g(\nabla_{\partial_k} \partial_t, \partial_l). \end{aligned}$$

Proof. The first equation is straightforward. The second follows from differentiating $g^{ik}g_{kl} = \delta_{il}$. $\square$

From the well-known formula for the derivative of the determinant,

$$(\det U)' = \det(U)\mathrm{tr}(U'),$$

we deduce:

Proposition 14. *The first variation of area is given by*

$$\frac{d}{dt}|\Sigma_t| = \int_\Sigma H_t v_t dA_t + \int_{\partial\Sigma} g(\nu_t, \frac{\partial f}{\partial t})dL_t.$$

Proof. Observe that

$$\begin{aligned}
\partial_t\sqrt{\det[g_{ij}]} &= \frac{1}{2}g^{ij}\partial_t g_{ij}\sqrt{\det[g_{ij}]} \\
&= g^{ij}g(\nabla_{\partial_i}\partial_t, \partial_j)\sqrt{\det[g_{ij}]} \\
&= (g^{ij}g(\nabla_{\partial_i}\partial_t^T, \partial_j) + g^{ij}g(\nabla_{\partial_i}N_t, \partial_j)v_t)\sqrt{\det[g_{ij}]} \\
&= (\mathrm{div}_{\Sigma_t}\partial_t^T + H_t v_t)\sqrt{\det[g_{ij}]}.
\end{aligned}$$

The first variation formula of area follows. $\square$

Next we look at the variations of the normal field.

Proposition 15.

$$\begin{aligned}
\nabla_{\partial_i}N_t &= g^{kl}B_{il}\partial_k, \\
\nabla_{\partial_t}N_t &= \nabla_{(\partial_t)^T}N_t - \nabla^{\Sigma_t}v_t.
\end{aligned}$$

where $\nabla^\Sigma v_t$ is the gradient of the function v_t on Σ_t.

Proof. Since $g(N_t, N_t) = 1$, $\nabla_{\partial_i}N_t$ and $\nabla_{\partial_t}N_t$ are tangent to Σ_t. The first equation is just the expression of $\partial_i N_t$ in the basis $\{\partial_i\}$. On the other hand, since $g(N_t, \partial_i) = 0$, we have

$$\partial_t N_t = g^{ik}g(\nabla_{\partial_t}N_t, \partial_k)\partial_i = -g^{ik}g(N_t, \nabla_{\partial_t}\partial_k)\partial_i = -g^{ik}g(N_t, \nabla_{\partial_t}\partial_k)\partial_i.$$

In local coordinates, the gradient of v_t in Σ_t is given by $\nabla^{\Sigma_t}v_t = (g^{ij}\partial_j v_t)\partial_i$. Then we have

$$g^{ik}g(N_t, \nabla_{\partial_k}(v_t N_t))\partial_i = g(N_t, N_t)(g^{ik}\partial_k v_t)\partial_i = \nabla^{\Sigma_t}v_t.$$

Therefore

$$\nabla_{\partial_t}N_t = \nabla_{(\partial_t)^T}N_t - \nabla^{\Sigma_t}v_t.$$

$\square$

Before we compute the variation of the mean curvature, let us recall the *Codazzi equation*:

$$g(R(U,V)N_t, W) = (\nabla^{\Sigma_t}_U B)(V,W) - (\nabla^{\Sigma_t}_V B)(U,W).$$

In this equation, R denotes the Riemann curvature tensor of (M, g) and U, V and W are tangent to Σ_t.

Taking $U = \partial_i$, $W = \partial_k$ and contracting, we obtain

$$Ric(V, N_t) = g^{ik}(\nabla^{\Sigma_t}_{\partial_i} B)(V, \partial_k) - dH_t(V).$$

for every V tangent to Σ_t.

Proposition 16. *The variation of the mean curvature is given by*

$$\partial_t H_t = dH_t(\partial_t^T) - L_{\Sigma_t} v_t.$$

where $L_{\Sigma_t} = \Delta_{\Sigma_t} + Ric(N_t, N_t) + |B_t|^2$ *is the Jacobi operator.*

Proof. Since $H_t = g^{ij} g(\nabla_{\partial_i} N_t, \partial_j)$,

$$\begin{aligned}
\partial_t H_t &= \partial_t g^{ij} g(\nabla_{\partial_i} N_t, \partial_j) + g^{ij} g(\nabla_{\partial_t} \nabla_{\partial_i} N_t, \partial_j) + g^{ij} g(\nabla_{\partial_i} N_t, \nabla_{\partial_t} \partial_j) \\
&= -2g^{ik} g^{jl} g(\nabla_{\partial_k} \partial_t, \partial_l) g(\nabla_{\partial_i} N_t, \partial_j) + g^{ij} g(R(\partial_t, \partial_i) N_t, \partial_j) \\
&\quad + g^{ij} g(\nabla_{\partial_i} \nabla_{\partial_t} N_t, \partial_j) + g^{ij} g(\nabla_{\partial_i} N_t, \nabla_{\partial_j} \partial_t) \\
&= -2g^{ik} g(\nabla_{\partial_k} \partial_t, \nabla_{\partial_i} N_t) - Ric(\partial_t, N_t) \\
&\quad + g^{ij} g(\nabla_{\partial_i} \nabla_{\partial_t} N_t, \partial_j) + g^{ij} g(\nabla_{\partial_i} N_t, \nabla_{\partial_j} \partial_t) \\
&= -g^{ij} g(\nabla_{\partial_i} N_t, \nabla_{\partial_j} \partial_t) - Ric(\partial_t, N_t) \\
&\quad + g^{ij} g(\nabla_{\partial_i} (\nabla_{\partial_t^T} N_t), \partial_j) - g^{ij} g(\nabla_{\partial_i} (\nabla^{\Sigma_t} v), \partial_j).
\end{aligned}$$

Now we use the contracted Codazzi equation:

$$\begin{aligned}
Ric(\partial_t^T, N_t) &= g^{ij} (\nabla^{\Sigma_t}_{\partial_i} B)(\partial_t^T, \partial_j) - dH(\partial_t^T) \\
&= g^{ij} \partial_i g(\nabla_{\partial_t^T} N_t, \partial_j) - g^{ij} g(\nabla_{(\nabla_{\partial_i} \partial_t^T)^T} N_t, \partial_j) \\
&\quad - g^{ij} g(\nabla_{\partial_t^T} N_t, (\nabla_{\partial_i} \partial_j)^T) - dH(\partial_t^T) \\
&= g^{ij} (\partial_i g(\nabla_{\partial_t^T} N_t, \partial_j) - g(\nabla_{\partial_t^T} N_t, \nabla_{\partial_i} \partial_j)) \\
&\quad - g^{ij} g(\nabla_{\partial_j} N_t, (\nabla_{\partial_i} \partial_t^T)^T) - dH(\partial_t^T) \\
&= g^{ij} g(\nabla_{\partial_i} (\nabla_{\partial_t^T} N_t), \partial_j) - g^{ij} g(\nabla_{\partial_j} N_t, \nabla_{\partial_i} \partial_t^T) - dH(\partial_t^T).
\end{aligned}$$

Hence, canceling out the corresponding terms, we have

$$\begin{aligned}
\partial_t H_t &= -g^{ij} g(\nabla_{\partial_i} N_t, \nabla_{\partial_j} N_t) v_t - Ric(N_t, N_t) v_t \\
&\quad + dH(\partial_t^T) - g^{ij} g(\nabla_{\partial_i} (\nabla^{\Sigma_t} v_t), \partial_j).
\end{aligned}$$

The formula follows. $\square$

Finally, we specialize the formulae above in the two particular cases we used in this paper. The proofs are immediate.

Proposition 17. *If* Σ_0 *is free boundary and* $(\partial_t)^T = 0$ *at* $t = 0$, *then*

$$(\partial_t H_t)|_{t=0} = -L_{\Sigma_0} v_0 \quad \textit{and} \quad \partial_t g(N_t, X)|_{t=0} = -\frac{\partial v_0}{\partial \nu_0} + g(N_0, \nabla_{N_0} X) v_0.$$

Proposition 18. *If each Σ_t is a constant mean curvature free boundary surface, then*

$$\partial_t H_t = -L_{\Sigma_t} v_t \quad \textit{and} \quad \frac{\partial v_t}{\partial \nu_t} = g(N_t, \nabla_{N_t} X) v_t.$$

References

[1] H. Bray, *The Penrose inequality in general relativity and volume comparison theorems involving scalar curvature*, Thesis, Stanford University (1997).

[2] H. Bray, S. Brendle and A. Neves, *Rigidity of area-minimizing two-spheres in three-manifolds*, Comm. Anal. Geom. 18 (2010), no 4., 821-830.

[3] M. Cai and G. Galloway, *Rigidity of area-minimizing tori in 3-manifolds of nonnegative scalar curvature*, Comm. Anal. Geom. 8 (2000), no 3., 565-573.

[4] J. Chen, A. Fraser and C. Pang, *Minimal immersions of compact bordered Riemann surfaces with free boundary*, arXiv:1209.1165.

[5] D. Fischer-Colbrie and R. Schoen, *The structure of complete stable minimal surfaces in 3-manifolds of nonnegative scalar curvature*, Comm. Pure Appl. Math. 33 (1980), no 2, 199-211.

[6] G. Huisken and S.-T. Yau, *Definition of center of mass for isolated physical systems and unique foliations by stable spheres with constant mean curvature*, Invent. Math. 124 (1996), no. 1-3, 281-311.

[7] J. Kazdan and F. Warner, *Prescribing curvatures*, Differential Geometry, Proc. Sympos. Pure Math., vol. 27, Amer. Math. Soc., Providence, R.I. (1975) 309-319.

[8] O. Ladyzhenskaia and N. Uralt'seva, *Linear and quasilinear elliptic equations*, Academic Press, New York (1968) 495 pp.

[9] M. Li, *Rigidity of area-minimizing disks in three-manifolds with boundary*, preprint.

[10] W. Meeks and S.T. Yau, *Topology of three-dimensional manifolds and the embedding problems in minimal surface theory*, Ann. of Math. (2) 112 (1980), no. 3, 441-484.

[11] W. Meeks and S.T. Yau, *The existence of embedded minimal surfaces and the problem of uniqueness*, Math. Z. 179 (1982), no. 2, 151-168.

[12] M. Micallef and V. Moraru, *Splitting of 3-Manifolds and rigidity of area-minimizing surfaces*, arXiv:1107.5346.

[13] I. Nunes, *Rigidity of area-minimizing hyperbolic surfaces in three-manifolds*, J. of Geom. Anal., published electronically 20 December 2011, doi: 10.1007/s12220-011-9287-8.

[14] R. Schoen and S.T. Yau, *Existence of incompressible minimal surfaces and the topology three dimensional manifolds with non-negative scalar curvature*, Ann. of Math (2) 110 (1979), no. 1, 127-142.

[15] L. Simon, *Lectures on geometric measure theory*, Proceedings of the Centre for Mathematical Analysis, Australian National University, Canberra, (1983) vii+272pp.

Instituto de Matemática Pura e Aplicada (IMPA), Estrada Dona Castorina 110, 22460-320 Rio de Janeiro, Brazil; lambroz@impa.br